

# TWO NEW DIFFERENT KINDS OF CONVEX DOMINATED FUNCTIONS AND INEQUALITIES VIA HERMITE-HADAMARD TYPE

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ABSTRACT. In this paper, we establish two new convex dominated function and then we obtain new Hadamard type inequalities related to this definitions.

## 1. INTRODUCTION

Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function and let  $a, b \in I$ , with  $a < b$ . The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known in the literature as Hadamard's inequality. Both inequalities hold in the reversed direction if  $f$  is concave.

In [1], Godunova and Levin introduced the following class of functions.

**Definition 1.** A function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is said to belong to the class of  $Q(I)$  if it is nonnegative and for all  $x, y \in I$  and  $\lambda \in (0, 1)$  satisfies the inequality;

$$(1.1) \quad f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}.$$

They also noted that all nonnegative monotonic and nonnegative convex functions belong to this class and also proved the following motivating result:

If  $f \in Q(I)$  and  $x, y, z \in I$ , then

$$(1.2) \quad f(x)(x-y)(x-z) + f(y)(y-x)(y-z) + f(z)(z-x)(z-y) \geq 0.$$

In fact (1.1) is even equivalent to (1.2). So it can alternatively be used in the definition of the class  $Q(I)$ .

In [2], Dragomir et al. defined the following new class of functions.

**Definition 2.** A function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is  $P$  function or that  $f$  belongs to the class of  $P(I)$ , if it is nonnegative and for all  $x, y \in I$  and  $\lambda \in [0, 1]$ , satisfies the following inequality;

$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y).$$

In [2], Dragomir et al. proved the following inequalities of Hadamard type for class of  $Q(I)$ – functions and  $P$ – functions.

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**Theorem 1.** *Let  $f \in Q(I)$ ,  $a, b \in I$  with  $a < b$  and  $f \in L_1[a, b]$ . Then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a} \int_a^b f(x)dx$$

and

$$\frac{1}{b-a} \int_a^b p(x) f(x)dx \leq \frac{f(a) + f(b)}{2}$$

where  $p(x) = \frac{(b-x)(x-a)}{(b-a)^2}$ ,  $x \in [a, b]$ .

**Theorem 2.** *Let  $f \in P(I)$ ,  $a, b \in I$  with  $a < b$  and  $f \in L_1[a, b]$ . Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x)dx \leq 2[f(a) + f(b)].$$

In [3] and [4], the authors connect together some disparate threads through a Hermite-Hadamard motif. The first of these threads is the unifying concept of a  $g$ -convex-dominated function. In [5], Hwang et al. established some inequalities of Fejér type for  $g$ -convex-dominated functions. Finally, in [6] Kavurmacı et al. introduced several new different kinds of convex-dominated functions and then gave H-H type inequalities for this classes of functions.

The main purpose of this paper is to introduce two new convex-dominated function and then present new H-H type inequalities related to these definitions.

## 2. $(g, Q(I))$ -CONVEX DOMINATED FUNCTIONS

**Definition 3.** *Let a nonnegative function  $g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  belong to the class of  $Q(I)$ . The real function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is called  $(g, Q(I))$ -convex dominated on  $I$  if the following condition is satisfied;*

$$(2.1) \quad \left| \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda} - f(\lambda x + (1-\lambda)y) \right| \leq \frac{g(x)}{\lambda} + \frac{g(y)}{1-\lambda} - g(\lambda x + (1-\lambda)y)$$

for all  $x, y \in I$  and  $\lambda \in (0, 1)$ .

The next simple characterisation of  $(g, Q(I))$ -convex dominated functions holds.

**Lemma 1.** *Let a nonnegative function  $g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  belong to the class of  $Q(I)$  and  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a real function. The following statements are equivalent:*

- (1)  $f$  is  $(g, Q(I))$ -convex dominated on  $I$ .
- (2) The mappings  $g - f$  and  $g + f$  are  $(g, Q(I))$ -convex on  $I$ .
- (3) There exist two  $(g, Q(I))$ -convex mappings  $l, k$  defined on  $I$  such that

$$f = \frac{1}{2}(l - k) \quad \text{and} \quad g = \frac{1}{2}(l + k) .$$

*Proof.*  $1 \iff 2$  The condition (2.1) is equivalent to

$$\begin{aligned} & g(\lambda x + (1-\lambda)y) - \frac{g(x)}{\lambda} - \frac{g(y)}{1-\lambda} \\ & \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda} - f(\lambda x + (1-\lambda)y) \\ & \leq \frac{g(x)}{\lambda} + \frac{g(y)}{1-\lambda} - g(\lambda x + (1-\lambda)y) \end{aligned}$$

for all  $x, y \in I$  and  $\lambda \in (0, 1)$ . The two inequalities may be rearranged as

$$(g+f)(\lambda x + (1-\lambda)y) \leq \frac{(g+f)(x)}{\lambda} + \frac{(g+f)(y)}{1-\lambda}$$

and

$$(g-f)(\lambda x + (1-\lambda)y) \leq \frac{(g-f)(x)}{\lambda} + \frac{(g-f)(y)}{1-\lambda}$$

which are equivalent to the  $(g, Q(I))$ -convexity of  $g+f$  and  $g-f$ , respectively.

$2 \iff 3$  Let we define the mappings  $f, g$  as  $f = \frac{1}{2}(l-k)$  and  $g = \frac{1}{2}(l+k)$ . Then if we sum and subtract  $f, g$ , respectively, we have  $g+f=l$  and  $g-f=k$ . By the condition 2 of Lemma 1, the mappings  $g-f$  and  $g+f$  are  $(g, Q(I))$ -convex on  $I$ , so  $l, k$  are  $(g, Q(I))$ -convex mappings on  $I$  too.  $\square$

**Theorem 3.** Let a nonnegative function  $g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  belong to the class of  $Q(I)$  and the real function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is  $(g, Q(I))$ -convex dominated on  $I$ . If  $a, b \in I$  with  $a < b$  and  $f, g \in L_1[a, b]$ , then one has the inequalities:

$$\left| \frac{4}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{4}{b-a} \int_a^b g(x) dx - g\left(\frac{a+b}{2}\right)$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b p(x) f(x) dx \right| \leq \frac{g(a)+g(b)}{2} - \frac{1}{b-a} \int_a^b p(x) g(x) dx$$

for all  $x, y \in I$  and  $p(x)$  as in Theorem 1.

*Proof.* By Definition 1 with  $\lambda = \frac{1}{2}$ ,  $x = ta + (1-t)b$ ,  $y = (1-t)a + tb$  and  $t \in [0, 1]$ , as the mapping  $f$  is  $(g, Q(I))$ -convex dominated function, we have that

$$\begin{aligned} & \left| 2[f(ta + (1-t)b) + f((1-t)a + tb)] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq 2[g(ta + (1-t)b) + g((1-t)a + tb)] - g\left(\frac{a+b}{2}\right). \end{aligned}$$

Integrating the above inequality over  $t$  on  $[0, 1]$ , the first inequality is proved.

Since  $f$  is  $(g, Q(I))$ -convex dominated using Definition 1 with  $x = a$ ,  $y = b$  and  $t \in [0, 1]$ , we can write

$$\begin{aligned} & |(1-t)f(a) + tf(b) - t(1-t)f(ta + (1-t)b)| \\ & \leq (1-t)g(a) + tg(b) - t(1-t)g(ta + (1-t)b) \end{aligned}$$

and

$$\begin{aligned} & |tf(a) + (1-t)f(b) - t(1-t)f((1-t)a + tb)| \\ & \leq tg(a) + (1-t)g(b) - t(1-t)g((1-t)a + tb). \end{aligned}$$

Then, adding above inequalities we have

$$\begin{aligned} & |[f(a) + f(b)] - t(1-t)[f(ta + (1-t)b) + f((1-t)a + tb)]| \\ & \leq [g(a) + g(b)] - t(1-t)[g(ta + (1-t)b) + g((1-t)a + tb)]. \end{aligned}$$

Integrating the resulting inequality over  $t$  on  $[0, 1]$ , we get the second inequality. The proof is completed.  $\square$

### 3. $(g, P(I))$ –CONVEX DOMINATED FUNCTIONS

**Definition 4.** Let a nonnegative function  $g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  belong to the class of  $P(I)$ . The real function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is called  $(g, P(I))$ –convex dominated on  $I$  if the following condition is satisfied;

$$\begin{aligned} (3.1) \quad & |[f(x) + f(y)] - f(\lambda x + (1-\lambda)y)| \\ & \leq [g(x) + g(y)] - g(\lambda x + (1-\lambda)y) \end{aligned}$$

for all  $x, y \in I$  and  $\lambda \in [0, 1]$ .

The next simple characterisation of  $(g, P(I))$ –convex dominated functions holds.

**Lemma 2.** Let a nonnegative function  $g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  belong to the class of  $P(I)$  and  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a real function. The following statements are equivalent:

- (1)  $f$  is  $(g, P(I))$ –convex dominated on  $I$ .
- (2) The mappings  $g - f$  and  $g + f$  are  $(g, P(I))$ –convex on  $I$ .
- (3) There exist two  $(g, P(I))$ –convex mappings  $l, k$  defined on  $I$  such that

$$f = \frac{1}{2}(l - k) \quad \text{and} \quad g = \frac{1}{2}(l + k) .$$

*Proof.*  $1 \iff 2$  The condition (3.1) is equivalent to

$$\begin{aligned} & g(\lambda x + (1-\lambda)y) - [g(x) + g(y)] \\ & \leq [f(x) + f(y)] - f(\lambda x + (1-\lambda)y) \\ & \leq [g(x) + g(y)] - g(\lambda x + (1-\lambda)y) \end{aligned}$$

for all  $x, y \in I$  and  $\lambda \in [0, 1]$ . The two inequalities may be rearranged as

$$(g + f)(\lambda x + (1-\lambda)y) \leq (g + f)(x) + (g + f)(y)$$

and

$$(g - f)(\lambda x + (1-\lambda)y) \leq (g - f)(x) + (g - f)(y)$$

which are equivalent to the  $(g, P(I))$ –convexity of  $g + f$  and  $g - f$ , respectively.

$2 \iff 3$  Let we define the mappings  $f, g$  as  $f = \frac{1}{2}(l - k)$  and  $g = \frac{1}{2}(l + k)$ . Then if we sum and subtract  $f, g$ , respectively, we have  $g + f = l$  and  $g - f = k$ . By the condition 2 of Lemma 2, the mappings  $g - f$  and  $g + f$  are  $(g, P(I))$ –convex on  $I$ , so  $l, k$  are  $(g, P(I))$ –convex mappings on  $I$  too.  $\square$

**Theorem 4.** Let a nonnegative function  $g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  belong to the class of  $P(I)$ . The real function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is  $(g, P(I))$ –convex dominated on  $I$ . If  $a, b \in I$  with  $a < b$  and  $f, g \in L_1[a, b]$ , then one has the inequalities:

$$\left| \frac{2}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{2}{b-a} \int_a^b g(x) dx - g\left(\frac{a+b}{2}\right)$$

and

$$\left| [f(a) + f(b)] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq [g(a) + g(b)] - \frac{1}{b-a} \int_a^b g(x) dx$$

for all  $x, y \in I$ .

*Proof.* By Definition 4 with  $\lambda = \frac{1}{2}$ ,  $x = ta + (1-t)b$ ,  $y = (1-t)a + tb$  and  $t \in [0, 1]$ , as the mapping  $f$  is  $(g, P(I))$ -convex dominated function, we have

$$\begin{aligned} & \left| [f(ta + (1-t)b) + f((1-t)a + tb)] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq [g(ta + (1-t)b) + g((1-t)a + tb)] - g\left(\frac{a+b}{2}\right). \end{aligned}$$

Integrating the above inequality over  $t$  on  $[0, 1]$ , the first inequality is proved.

Since  $f$  is  $(g, P(I))$ -convex dominated using Definition 4 with  $x = a$ ,  $y = b$  and  $t \in [0, 1]$ , we can write

$$\begin{aligned} & |[f(a) + f(b)] - f(ta + (1-t)b)| \\ & \leq g(a) + g(b) - g(ta + (1-t)b). \end{aligned}$$

Integrating the above resulting inequality over  $t$  on  $[0, 1]$  we get the second inequality. The proof is completed.  $\square$

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